Joint reality and Bell inequalities for consecutive measurements

R. Lapiedra(*)

Department of Astronomy and Astrophysics, University of Valencia, 46100 Burjassot (Valencia), Spain

PACS. 03.65.Ud – Entanglement and quantum non-locality.
PACS. 03.65.Ta – Foundations of Quantum Mechanics, measurement theory.

Abstract. – Some new Bell inequalities for consecutive measurements are deduced under joint realism assumption, using some perfect correlation property. No locality condition is needed. When the measured system is a macroscopic system, joint realism assumption substitutes the non-invasive measurability hypothesis advantageously, provided that the system satisfies the perfect correlation property. The new inequalities are violated quantically. This violation can be expected to be more severe than in the case of precedent temporal Bell inequalities. Some microscopic and mesoscopic situations, in which the new inequalities could be tested, are roughly considered.

1. Introduction. – Besides the ordinary Bell inequalities [1, 2] for entangled systems, there also exist so-called temporal Bell inequalities [3-5] for a single system. In the seminal paper of Leggett and Garg [6], the authors consider a macroscopic system and make two general assumptions:

i) Macroscopic realism: "A macroscopic system with two or more macroscopically distinct states available to it will at all times be in one or the other of these states".

ii) Noninvasive measurability (NIM): "It is possible, in principle, to determine the state of the system with arbitrarily small perturbation in its subsequent dynamics".

With these two assumptions, these authors prove some temporal Bell inequalities for such a macroscopic system, where the measurement times, t_i, play the role of the polarizer settings in the ordinary Bell inequalities. NIM assumption is obviously not valid for quantum systems, and has been criticized for macroscopic systems [7]. In spite of these criticisms, it seems that the idea of an *ideal negative experiment*, or alternatively the *coupling of the system to a microscopic probe*, as explained in [6], can change NIM into a reasonable assumption.

Whatever it be, the main purpose of the present paper is to prove some new Bell inequalities for consecutive measurements, retaining the realism of assumption i), but changing the NIM for a new assumption, that encompass the above assumption i), and becomes extremely natural and plausible if NIM is assumed, but not necessarily the reverse way. We will call this new assumption the *joint reality* assumption and we will state it below. Contrarily to what happens

(*) E-mail address: ramon.lapiedra@uv.es (R. Lapiedra). Telephone: 96 3543077

in the temporal Bell inequalities of Leggett and Garg, we here deal again with the original polarizer settings, instead of the above measurement times. Thus, in our Bell inequalities, we will perform different kinds of measurements in *a time ordered way*, on a non extended system.

The new Bell inequalities for consecutive measurements, which apply to any macroscopic or microscopic system, will be valid provided that the above assumption holds and the system obeys a so called perfect correlation property. This property is always valid for quantum systems and could be valid for macroscopic systems. In any case, for these last systems, one can always test whether the property is actually satisfied or not.(This perfect correlation property will be properly described below. Then, it will become clear that its validity is a first indication of the possible correctness of the original NIM assumption). The new Bell inequalities will be violated by quantum mechanics.

We will now go on to state this *joint reality* assumption, which will substitute the above two assumptions i) and ii). Consider an ensemble of systems S, prepared in some way at an initial time. S can be either macroscopic or not, and has a dichotomic magnitude, M, that is, a magnitude which only takes two values, say ± 1. We will measure M for three different values, a, b, and c, of an external parameter. (An example could be a one half spin particle measured on three different directions). Then, the above joint reality assumption assumes the joint existence of a reality behind any obtainable measurement outcome. More precisely we will denote this joint reality by $(a^\alpha b^\beta c^\gamma)$, where α, β, $\gamma = \pm$. That is, $(a^\alpha b^\beta c^\gamma)$ is the reality such that, if we took a measurement above, for the parameter value a, we would obtain $\alpha 1$, i.e., $+1$ or -1, according to the value of α. Similar for the other *directions* b and c. Notice that, like d'Espagnat in Ref. [8], and Wigner in Ref. [9], we assume the existence of a reality for all the possible results of all possible measurements, even if each actual measurement is taken in a single randomly chosen direction. As we have commented above, this kind of reality is a very natural assumption as far as one assumes NIM. We will comment below why it will be also interesting to assume initially this kind of reality in the case of a quantum system, where obviously the NIM assumption is non valid.

On the other hand, suppose we perform two immediately consecutive measurements for the same external parameter value. Then, we assume that the corresponding outcomes are *perfectly correlated*, i. e., if the first measurement value is $+1$, the second one is always $+1$, and likewise for a value of -1. This will be called the *perfect correlation* property. Obviously, this property is always satisfied when S is a quantum system, a *qubit* in this paper (i.e., a quantum system whose space of states is 2-dimensional), and we take pairs of immediately consecutive measurements for different *directions* randomly chosen among the same three different external parameter values. For a macroscopic system, perfect correlation property can be expected to be valid as far as the NIM assumption is correct. But, here, the question is simply whether experience will show or not that the property is satisfied. If it does, we meet the conditions to prove our inequalities. Otherwise, we could not prove them.

Thus, under joint realism assumption, using the perfect correlation property, we will prove some new Bell inequalities where the measurement times, t_i, of Ref. [6] will be replaced by the above parameter values, a, b, and c. In Ref. [10] a similar problem is addressed. The authors prove the inequalities for consecutive measurements which are analogous to the ordinary CHSH inequalities [2], under the "locality in time" assumption. We will not use here this doubtful assumption, which will be replaced by the above joint reality assumption and the use of the perfect correlation property.

2. Proving the new Bell inequalities for consecutive measurements. – Let us consider the above perfectly correlated system, S, with its dichotomic magnitude, M, measured randomly

and consecutively for the external parameter values a, b, and c. We want to prove some Bell inequality for the outcomes of these measurements, assuming joint realism and using perfect correlation. In order to do so, we will adapt to our case a proof of an ordinary Bell inequality for a singlet-state pair of entangled qubits. Here we adapt this proof in the form given by d'Espagnat [8], even if the proof was first given by Wigner [9].

Let us be more precise about the kind of experiment we are going to consider. In each system, S, of the above ensemble, we take two immediately consecutive measurements of M for two independent values, randomly chosen, of the three fixed external parameter values a, b, and c. We will call each of these pairs of measurements a *run*. Then, as we have explained before, we assume the existence of a joint reality behind any obtainable measurement outcome. Let us consider the number of these joint realities $(a^\alpha b^\beta c^\gamma)$, which are present *after the first measurement of every run and before the second measurement.* Let us denote this number by $N(a^\alpha b^\beta c^\gamma)$. We will define

$$N(a^+b^-) \equiv N(a^+b^-c^+) + N(a^+b^-c^-) \tag{1}$$

$$N(a^+c^-) \equiv N(a^+b^+c^-) + N(a^+b^-c^-) \tag{2}$$

$$N(b^+c^-) \equiv N(a^+b^+c^-) + N(a^+b^-c^-) \tag{3}$$

From this, we readily have:

$$N(a^+c^-) \leq (N(a^+b^-) + N(b^+c^-)). \tag{4}$$

Now, let us consider, for example, the number of *runs*, $N[a^+b^-]$, where a^+ is the outcome of the first measurement and b^- the outcome of the second one. (Notice that we use square brackets for measurement outcomes and standard brackets for hypothetical realities). Obviously, these *runs* can only come from the above realities (b^-) between the first and the second measurement. Furthermore, from the perfect correlation assumption, they can only come from the more specific realities (a^+b^-). (The notation (b^-) and (a^+b^-) should be obvious). Then, given a reality between both measurements such as (a^+b^-), what is the probability of obtaining a *run* like $[a^+b^-]$? Since the choice of any one of the three parameters, a, b, c, is a random choice, this probability is just 1/9. This means that we can write

$$N(a^+b^-) = 9N[a^+b^-], \tag{5}$$

and similarly for $N[a^+c^-]$ and $N[b^+c^-]$. Thus, taking into account Eq. (4), we obtain the temporal Bell inequality:

$$N[a^+c^-] \leq N[a^+b^-] + N[b^+c^-], \tag{6}$$

for the observable quantities $N[a^+c^-]$, $N[a^+b^-]$ and $N[b^+c^-]$.

Notice that in this proof it is essential to define the above joint reality $(a^\alpha b^\beta c^\gamma)$ as the joint reality which is present before the second measurement of the *run* and after the first one. In this way, the reality can only be changed by the second measurement. But this change is irrelevant to the completion of our proof, since in a *run* we do not consider a third measurement.

If one prefers to speak in terms of probabilities corresponding to the numbers in inequality (6), we can write this inequality as

$$P(a^+, c^-) \leq (P(a^+, b^-) + P(b^+, c^-)), \tag{7}$$

and in a similar way

$$P(a^-, c^+) \leq (P(a^-, b^+) + P(b^-, c^+)). \tag{8}$$

Or, in terms of the expected value,

$$E(a, b) = P(a^+, b^+) + P(a^-, b^-) - P(a^+, b^-) - P(a^-, b^+), \tag{9}$$

taking into account inequalities (7) and (8), we obtain:

$$E(a, b) + E(b, c) - E(a, c) \leq 1. \tag{10}$$

At first sight, one might think that inequalities (7), (8), or (10) are of no interest since, if they were experimentally violated, this could always be explained by some transmission of information between the two consecutive measurements of the *run*. But this is not true since, as we have seen, inequalities (7), (8), and (10) have been deduced from the joint realism assumption, using the perfect correlation property, without any further assumptions. Therefore, we can transmit all kinds of information we want between both measurements, but if perfect correlation and joint realism are preserved, as we assume, inequalities (7), (8), and (10) must remain true.

Now, we could find that the joint realism assumed in the present paper is a too restrictive postulate in the case of a quantum system, and, thus, a non convincing postulate for such a system. In fact, in quantum mechanics, two non commuting observables cannot be measured at the same time. Furthermore, the orthodox interpretation of the theory assumes that it is not only that we cannot jointly measure them, but it asserts that the corresponding joint reality does not exist. On the other hand, as we will see in the next Section, quantum mechanics entails the violation of our Bell inequalities. Then, we can say that the non existence of joint reality in the case of a qubit is not a question of interpretation, but a prediction of quantum mechanics which could be easily tested experimentally, by testing these Bell inequalities. Obviously, it is to be expected that experience will agree in this point with quantum mechanics and so that it will reject the assumed joint reality.

3. Quantum violation of the new Bell inequalities . – Let us assume that our system S is a qubit. Then, a normalized general state, $|\psi>$, can be written as:

$$|\psi >= s|e+> +(1-s^2)^{1/2} e^{i\phi}|e->, \tag{11}$$

where $|e+>$ and $|e->$ are the eigenstates of eigenvalues ± 1, respectively, for a given "direction" e. Since for any "direction" x the corresponding eigenstates, $|x+>$ and $|x->$, are orthogonal unit vectors in a 2-dimensional Hilbert space, it is straightforward to show that an angle α_x and a phase ϕ, always exist such that

$$|x+ >= [(1+\cos\alpha_x)/2]^{1/2}|e+> +[(1-\cos\alpha_x)/2]^{1/2} e^{i\phi}|e->, \tag{12}$$

$$|x- >= [(1-\cos\alpha_x)/2]^{1/2}|e+> -[(1+\cos\alpha_x)/2]^{1/2} e^{i\phi}|e-> . \tag{13}$$

This means, as it is well-known, that x and e can always be interpreted as two unit 3-vectors in $\mathbf{R}^3$, $\mathbf{x}$ and $\mathbf{e}$, respectively, which appear in these equations only through their 3-scalar product $\mathbf{x.e} = \cos\alpha_x$.

Hence, when measuring the above dichotomic magnitude M for the three external parameter values, a, b, and c, we can always say that these measurements have been taken for the corresponding unit 3-vectors, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

Let us consider the different probabilities, $P(\mathbf{a}^{\pm}, \mathbf{b}^{\pm})$, of obtaining ± 1 for the two consecutive measurements of the *runs* where chance has selected, respectively, the unit 3-vectors **a** and **b**. After some basic algebra, we find

$$P(\mathbf{a}^{+}, \mathbf{b}^{-}) = s^2(1 - \mathbf{a}.\mathbf{b})/2, P(\mathbf{a}^{-}, \mathbf{b}^{+}) = (1 - s^2)(1 - \mathbf{a}.\mathbf{b}/2. \tag{14}$$

Thus, according to Eq. (9), we obtain:

$$E(\mathbf{a}, \mathbf{b}) = \mathbf{a}.\mathbf{b}, \tag{15}$$

which differs in sign from the similar result for the expected value in the case of an entangled pair of qubits in the singlet state. (Obviously, for $E(a, c)$ and $E(c, b)$, we have similar equations to Eq. (15)).

Notice that the result (15) has the remarkable property of being independent of the initial state of the particle [10], that is, in (15), $E(\mathbf{a}, \mathbf{b})$ does not depend on s or ϕ appearing in Eq. (11), while$P(\mathbf{a}^{\pm}, \mathbf{b}^{\pm})$ does depend on s. All this means that the version (10) of our Bell inequalities does not depend on the initial state of the system S, while versions (7) or (8) do.

Bearing in mind Eq. (15) and the similar ones, the Bell inequality (10) becomes

$$\mathbf{a}.(\mathbf{b} - \mathbf{c}) + \mathbf{b}.\mathbf{c} \leq 1, \tag{16}$$

which is clearly violated by any two orthogonal unit 3-vectors **b** and **c**, if the unit 3-vector **a** is collinear to $\mathbf{b} - \mathbf{c}$. In this case, the left hand side of inequality (16) takes the value $\sqrt{2}$.

Similarly one can see that the quantum mechanics of qubit (11) violates inequalities (7) or (8), but this violation depends on the initial state of the qubit. For example, if this initial state is the eigenstate $|a+>$, for the different probabilities appearing in inequality (7) one finds:

$$P(a^{+}, c^{-}) = (1 - \mathbf{a}.\mathbf{c})/2, P(a^{+}, b^{-}) = (1 - \mathbf{a}.\mathbf{b})/2, P(b^{+}, c^{-}) = (1 + \mathbf{a}.\mathbf{b})(1 - \mathbf{b}.\mathbf{c})/4. \tag{17}$$

Then, for inequality (7) we get:

$$\mathbf{b}.(\mathbf{a} + \mathbf{c}) - 2\mathbf{a}.\mathbf{c} + (\mathbf{a}.\mathbf{b})(\mathbf{b}.\mathbf{c}) \leq 1. \tag{18}$$

For $\mathbf{a}.\mathbf{c} = 0$ and $\mathbf{b} = (\mathbf{a} + \mathbf{c})/\sqrt{2}$, this inequality is more severely violated than the above inequality (16), since the left hand side becomes $\sqrt{2} + 1/2$ for this configuration, instead of the above $\sqrt{2}$ for inequality (16). This means that inequalities (7) or (8) are more severely violated than inequality (10), which is in fact our version of the original inequalities of Leggett and Garg. Thus, in the macroscopic domain, we expect that our inequalities (7) or (8) could be more severely violated than Leggett and Garg inequalities.

4. Some examples. – Once we have seen that the new Bell inequalities (7), (8), and (10) can be violated by quantum mechanics, we roughly turn to the question of how this violation could be experimentally produced. Here, the problem is that we need to perform two successive measurements on the same system, and not merely in two different parts of the same system, as in the ordinary space entangled Bell inequalities. Then, in the quantum case, we must guarantee that the first of these two measurements is always a first class measurement, that is, a preparation-like measurement, in order to preserve the existence of the system and then be able to take the second measurement. These conditions can be fulfilled, in principle, in the case where the measured system is a one half spin particle, whose spin is successively measured in different directions, with a Stern-Gerlach device. We must then distinguish two

cases, according to whether we want to test inequality (10), or alternatively one of the two inequalities (7) or (8).

In the first case, in order to obtain the measurement outcomes to calculate, for example, the expected value $E(a,b)$ in (10), we must perform two different measurement series, two *run* series to be more precise (following a similar strategy to the one stated in [6], where the authors combine different ideal negative-result setups). One *run* series to obtain the probabilities $P(a^+, b^+)$ and $P(a^+, b^-)$ and the other *run* series to obtain the probabilities $P(a^-,b^-)$ and $P(a^-,b^+)$. In the first series we only retain the a^+ outcomes corresponding to a preparation-like measurement. In this way, the spin particle is still available for a second measurement in direction b. We will proceed in a similar way for the second series, where in another large and identical ensemble we will only retain the a^- outcomes corresponding to another preparation-like measurement. In this way, we will be able to measure the three expected values $E(a,b)$, $E(b,c)$ and $E(a,c)$, and thus to test inequality (10). Notice that, as we have already remarked, in the present case we do not need to prepare the system in any particular state before each *run*.

Nevertheless, it could be doubted whether this method, of performing two different series of measurements, can guarantee the existence of a "common probability space", as argued in Ref. [11]. Then, to overcome this unfair state, we can consider the second case: the case corresponding to, let us say, inequality (7), which is, furthermore, an interesting case in itself. In this second case, we assume that the a^+ and b^+ outcomes in (7), related to the first measurements of each *run*, refer to preparation-like measurements. In the present case, the different probabilities which appear in (7) do depend on the particle state before each run. Then, for each run, we will prepare this previous state as an a^+ eigenstate. This is what has been assumed in order to deduce the inequality (18) that, as we have seen, is more severely violated than the corresponding inequality (16). In all, for each run, we must perform three successive Stern-Gerlach measurements: first, we must prepare the a^+ eigenstate, and then perform two successive measurements from this eigenstate. In this way we will be able to measure the three probabilities of inequality (7) and thus to test this inequality.

Let us emphasize that, as requested in Ref. [11], when testing inequality (7), there is a "common probability space" behind inequalities (7) or (8), though this is simply forced by joint reality assumption, which by itself means the common existence of different settings for the same event of the sample space.

Thus, since we can scarcely doubt that inequality (7), if tested, would be experimentally violated, we must conclude that joint realism is ruled out, not by any quantum mechanics interpretation, but by quantum mechanics itself. As we have already commented, this would be so, without any concern about locality conditions, since the proof of the new Bell inequalities, we are considering here, do not rely on the locality assumption (no locality loophole can be present here). Furthermore, according to what we have just explained, the existence of a common probability space would also be guarantied.

On the other hand, in the case of macroscopic systems, the problem of the possible measurements which destroy the system is absent. So, the problem of having more than a single probability space is also absent, by two different reasons. First of all, because it is forced by the joint reality assumption. Also, because now, in the case of inequality (10), differently to what happens in the microscopic case, we do not need to rely on the strategy of two different measurement series.

Finally, one could consider a micrometer sized super-conducting loop, with Josephson junctions [12], to test our Bell inequalities in the macroscopic case, as it has been considered by several authors [7]. But it seems to us that it would be more interesting to use our temporal Bell inequalities to test realism in the case of mesoscopic dichotomic random systems whose

randomness could be not so obviously retraced to an enclosed quantum system.

6.-Conclusions. – In the present paper, we have proved some Bell inequalities for consecutive measurements under the assumptions of "joint realism", using the perfect correlation property, for any kind of physical system, macroscopic or microscopic, with a randomly dichotomic magnitude, i.e., a magnitude which randomly takes two values. The measurement outcomes are the response of the system to some different external parameter values, as in the standard Bell inequalities. These different parameter values play the role of the different measurement times in the seminal paper of Legget and Garg [6]. In the paper, these authors deal with realism and NIM assumptions in the context of macroscopic systems. In the present paper, we deal jointly with macroscopic or microscopic systems, by substituting both assumptions for the joint reality assumption, and by using the perfect correlation property. Contrarily to the case of NIM assumption, joint reality can be asserted, in principle, either for microscopic systems or for macroscopic ones.

On the other hand**,** the perfect correlation property is always verified by quantum systems. When the physical system is a macroscopic one, one must verify whether the perfect correlation property is satisfied. One can expect that this property will be verified in the macroscopic case on the grounds of the joint reality assumption**.**

The new assumption of joint reality substitutes NIM assumption advantageously because that joint reality assumption can be applied, in principle, to quantum systems too, and also because it has provided us with new Bell inequalities for consecutive measurements, which can be expected to be more severely violated than the temporal ones from Leggett and Garg. Then, notice that, when trying to prove our Bell inequalities, if joint reality is assumed, and perfect correlation holds, we do not need to be concerned with any kind of information which could be propagated between the two measurements of a *run.*

* * *

This work has been supported by the Generalitat Valenciana GV05/264.
I am grateful to E.Santos for fruitful discussions.

REFERENCES

[1] BELL J. S., *Physics* **1** (1964) 195.
[2] ClAUSER, J. F., HORNE M. A., SHIMONY A. and HOLT R. H., *Phys. Rev. Lett.* **23** (1969) 880.
[3] PAZ J. P. and MAHLER G., *Phys. Rev. Lett.* **71** (1993) 3235.
[4] HUELGA S. F., MARSHALL T. W. and SANTOS E., *Phys. Rev. A* **54** (1996) 1798.
[5] HUELGA S. F., MARSHALL T. W. and SANTOS E., *Europhys. Lett.* **38** (1997) 249.
[6] LEGGETT A. J. and GARG A., *Phys. Rev. Lett.* 54 (1985) 857.
[7] BALLENTINE L. E., *Phys. Rev. Lett.*, **59** (1987) 1493 ; PERES A., *Phys. Rev. Lett.,* **61** (1988) 2019; CALARCO T. and ONOFRIO R., *Phys. Lett. A*, **198** (1995) 279; **208** (1995) 40; BENATTI F., GHIRARDI G. C. and GRASSI R., *Nuovo Cimento B*, **110** (1995) 593. See also the two responses of Leggett and Garg: LEGGETT A. J. and GARG A., *Phys. Rev. Lett.* **59** (1987) 1621; **63** (1989) 2159.
[8] D'ESPAGNAT B., *Scientific American* **241,** num. 5 (1979) 158.
[9] WIGNER E. P., *American Journal of Phys.* **38** (1970) 1005.
[10] BRUKNER Č., TAYLOR S., CHEUNG S. and VEDRAL V., *arXiv.quant-ph/0402127 v1.*
[11] HESS K. and PHILLIP W., *Foundations of Physics*35 (2005) 1749.
[12] VAN DER WAL C. *et al*, *Science*, **290** (2000) 773